

CAUCHY PROBLEM FOR SEMILINEAR WAVE EQUATION WITH TIME-DEPENDENT METRICS

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ABSTRACT. We establish the existence of weak solutions u of the semilinear wave equation $\partial_t^2 u - \operatorname{div}_x(a(t, x)\nabla_x u) = f_k(u)$ where $a(t, x)$ is equal to 1 outside a compact set with respect to x and a non-linear term f_k which satisfies $|f_k(u)| \leq C|u|^k$. For some non-trapping time-periodic perturbations $a(t, x)$, we obtain the long time existence of solution for small initial data.

1. INTRODUCTION

Consider the semilinear Cauchy problem

$$\begin{cases} u_{tt} - \operatorname{div}_x(a(t, x)\nabla_x u) - f_k(u) = 0, & (t, x) \in \mathbb{R}^{n+1}, \\ (u, u_t)(0, x) = (g_1(x), g_2(x)) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where for a given $k > 1$ the non-linearity f_k is assumed to be a C^1 function on \mathbb{R} satisfying $f_k(0) = 0$, $|f'_k(u)| \leq C|u|^{k-1}$ and the perturbation $a(t, x) \in C^\infty(\mathbb{R}^{n+1})$ satisfies the conditions:

$$\begin{aligned} (i) & \quad C_0 \geq a(t, x) \geq c_0 > 0, \quad \forall (t, x) \in \mathbb{R}^{n+1}, \\ (ii) & \quad \text{there exists } \rho > 0 \text{ such that } a(t, x) = 1 \text{ for } |x| \geq \rho. \end{aligned} \quad (1.2)$$

Denote by $\dot{H}^1(\mathbb{R}^n)$ the closure of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|\varphi\|_{\dot{H}^1} = \left(\int_{\mathbb{R}^n} |\nabla \varphi(x)|^2 dx \right)^{\frac{1}{2}}.$$

Throughout this paper we assume that $n \geq 3$ and that the initial data g is in the energy space $\dot{\mathcal{H}}_1(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Consider the linear problem associated to (1.1)

$$\begin{cases} u_{tt} - \operatorname{div}_x(a(t, x)\nabla_x u) = 0, & (t, x) \in \mathbb{R}^{n+1}, \\ (u, u_t)(s, x) = (g_1(x), g_2(x)) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.3)$$

where $g \in \dot{\mathcal{H}}_1(\mathbb{R}^n)$. The solution of (1.3) is given by the propagator

$$\mathcal{U}(t, s) : \dot{\mathcal{H}}_1(\mathbb{R}^n) \ni (g_1, g_2) = g \mapsto \mathcal{U}(t, s)g = (u, u_t)(t, x) \in \dot{\mathcal{H}}_1(\mathbb{R}^n).$$

We denote by $U(t, s)$ and $V(t, s)$ the operators defined by

$$\begin{aligned} U(t, s)f &= (\mathcal{U}(t, s)(f, 0))_1, \quad f \in \dot{H}^1(\mathbb{R}^n), \\ V(t, s)h &= (\mathcal{U}(t, s)(0, h))_1, \quad h \in L^2(\mathbb{R}^n), \end{aligned}$$

where $(h_1, h_2)_1 = h_1$. We say that $u \in \mathcal{C}([0, T_1], \dot{H}^1)$ is a weak solution of (1.1) if for all $t \in [0, T_1]$ we have

$$\begin{aligned} u(t) &= \left(\mathcal{U}(t, 0)g + \int_0^t \mathcal{U}(t, s)(0, f_k(u(s))) ds \right)_1 \\ &= (\mathcal{U}(t, 0)g)_1 + \int_0^t V(t, s)(f_k(u(s))) ds. \end{aligned} \quad (1.4)$$

Let $a(t, x) = 1$. Then we have the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta_x u - f_k(u) = 0, & (t, x) \in \mathbb{R}^{n+1}, \\ (u, u_t)(0, x) = (g_1(x), g_2(x)) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (1.5)$$

The problem (1.5) has been extensively studied for $g \in \dot{\mathcal{H}}_1(\mathbb{R}^n)$. For example, the global well-posedness of the problem (1.5) has been established for the case of the sub-critical growth $1 < k < 1 + \frac{4}{n-2}$ (see [6] and [19]) or for the case of the critical growth $k = 1 + \frac{4}{n-2}$ (see [15] and [19]). For the case $k > 1 + \frac{4}{n-2}$ it is not yet clear whether there exists or not a global regular solution for the Cauchy problem (1.5) with arbitrary initial data. On the other hand, local well posedness as well as global well-posedness, with small initial data in fractional Sobolev spaces have been also studied by many authors for the problem (1.5) under minimal regularity assumptions on the initial data (see [8] and [19]).

In [18] Michael Reissig and Karen Yagdjian established Strichartz decay estimates for the solution of strictly hyperbolic equations of second order with coefficients depending only on t . We can apply these estimates to prove existence results for the solution of problem (1.1) when $a(t, x) = a(t)$ is independent on x (see [11] and [20] for the case of the free wave equation). It seems that our paper is one of the first works where one treats non-linear wave equations with time dependent perturbations $a(t, x)$ depending on t and x .

The goal of this paper is to find sufficient conditions for the existence of a weak solution of (1.1) when $0 \leq t \leq T_1$. For this purpose, we will use Strichartz estimates to study local and long time existence and uniqueness of solutions of the problem (1.1). In fact, for suitable k Strichartz estimates allow us to find a fixed point of the map

$$\mathcal{G}(u) = (\mathcal{U}(t, 0)g)_1 + \int_0^t V(t, s)f_k(u(s))ds,$$

in $\mathcal{C}([0, T_1], \dot{H}^1)$ for some well chosen $k > 1$. The fixed point of \mathcal{G} is local weak solution of (1.1). In [9] we have established local homogeneous Strichartz estimates for $n \geq 3$ and $a(t, x)$ satisfying (1.2), and global homogeneous Strichartz estimates when $n \geq 3$ is odd for some non-trapping time-periodic perturbation $a(t, x)$ (see Section 2). Recently global Strichartz estimates for even dimensions $n \geq 4$ have been obtained in [10]. One way to obtain global weak solutions is to apply global non homogeneous Strichartz estimates concerning the solution of the Cauchy problem for $u_{tt} - \operatorname{div}(a(t, x)\nabla u(x)) = G(t, x)$. This leads to some difficulties and this case is not covered by our results in [9] and [10]. On the other hand, for time dependent perturbations we have no conservation laws. For these reasons we obtain only long time existence of weak solution in Section 4. In Section 2 we recall the estimates for the linear wave equation with metric $a(t, x)$. In Section 3 we obtain local existence results, while in Section 4 we deal with long time existence.

Remark 1. Let the metric $(a_{ij}(t, x))_{1 \leq i, j \leq n}$ be such that for all $i, j = 1 \cdots n$ we have

- (i) there exists $\rho > 0$ such that $a_{ij}(t, x) = \delta_{ij}$, for $|x| \geq \rho$, with $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$,
- (ii) there exists $T > 0$ such that $a_{ij}(t + T, x) = a_{ij}(t, x)$, $(t, x) \in \mathbb{R}^{n+1}$,
- (iii) $a_{ij}(t, x) = a_{ji}(t, x)$, $(t, x) \in \mathbb{R}^{n+1}$,
- (iv) there exist $C_0 > c_0 > 0$ such that $C_0|\xi|^2 \geq \sum_{i,j=1}^n a_{ij}(t, x)\xi_i\xi_j \geq c_0|\xi|^2$, $(t, x) \in \mathbb{R}^{1+n}$, $\xi \in \mathbb{R}^n$.

If we replace $a(t, x)$ in (1.1) we get the following problem

$$\begin{cases} u_{tt} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(t, x) \frac{\partial}{\partial x_j} u \right) - f_k(u) = 0, & (t, x) \in \mathbb{R}^{n+1}, \\ (u, u_t)(s, x) = (f_1(x), f_2(x)) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Since, with the same conditions as (1.1), global Strichartz estimates are true for solutions of this equation when $f_k(u) = 0$ (see [10]), all the results of this paper remain true for this problem.

2. STRICHARTZ ESTIMATES FOR THE LINEAR EQUATION

In this section we recall some results concerning Strichartz estimates for the problem (1.3). We suppose that $a(t, x)$ satisfies the conditions (1.2). It was established in [9] that we have the following estimates.

Theorem 1. Assume $n \geq 3$ and let $a(t, x)$ be a C^∞ function on \mathbb{R}^{n+1} satisfying conditions (1.2). Let $2 \leq p, q < +\infty$, $\gamma > 0$ be such that

$$\frac{1}{p} = \frac{n(q-2)}{2q} - \gamma, \quad \frac{1}{p} \leq \frac{(n-1)(q-2)}{4q}. \quad (2.1)$$

Then there exists $\delta > 0$ such that for the solution $u(t, x)$ of (1.3) with $s = 0$ we have

$$\|u\|_{L^p([0, \delta], L^q(\mathbb{R}_x^n))} + \|u(t)\|_{C([0, \delta], \dot{H}^\gamma(\mathbb{R}_x^n))} + \|\partial_t(u)(t)\|_{C([0, \delta], \dot{H}^{\gamma-1}(\mathbb{R}_x^n))} \leq C(p, q, \rho, n) \|g\|_{\mathcal{H}_\gamma}. \quad (2.2)$$

Now, let $a(t, x)$ be T -periodic with respect to t which means

$$a(t+T, x) = a(t, x), \quad \forall (t, x) \in \mathbb{R}^{n+1}.$$

Moreover, we impose two hypothesis. The first one says that the perturbation $a(t, x)$ is non-trapping. More precisely, consider the null bicharacteristics $(t(\sigma), x(\sigma), \tau(\sigma), \xi(\sigma))$ of the principal symbol $\tau^2 - a(t, x)|\xi|^2$ of $\partial_t^2 - \operatorname{div}(a \nabla_x u)$ satisfying

$$t(0) = 0, |x(0)| \leq \rho, \quad \tau^2(\sigma) = a(t(\sigma), x(\sigma))|\xi(\sigma)|^2.$$

We introduce the following condition.

(H1) We say that the metric $a(t, x)$ is non-trapping if for each $R > \rho$ there exists $S_R > 0$ such that

$$|x(\sigma)| > R \text{ for } |\sigma| \geq S_R.$$

Notice that if we have trapping metrics, there exist solutions of (1.3) whose local energy is exponentially growing (see [2]). Thus for trapping metrics it is not possible to establish global Strichartz estimates.

Let $\psi_1, \psi_2 \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We define the cut-off resolvent associated to problem (1.3) by $R_{\psi_1, \psi_2}(\theta) = \psi_1(\mathcal{U}(T, 0) - e^{-i\theta})^{-1}\psi_2$. Consider the following assumption.

(H2) Let $\psi_1, \psi_2 \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi_i = 1$ for $|x| \leq \rho + 1 + 3T, i = 1, 2$. Then the operator $R_{\psi_1, \psi_2}(\theta)$ admits a holomorphic extension from $\{\theta \in \mathbb{C} : \operatorname{Im}(\theta) \geq A > 0\}$ to $\{\theta \in \mathbb{C} : \operatorname{Im}(\theta) \geq 0\}$, for $n \geq 3$, odd, and to $\{\theta \in \mathbb{C} : \operatorname{Im}(\theta) > 0\}$ for $n \geq 4$, even. Moreover, for n even, $R_{\psi_1, \psi_2}(\theta)$ admits a continuous extension from $\{\theta \in \mathbb{C} : \operatorname{Im}(\theta) > 0\}$ to $\{\theta \in \mathbb{C} : \operatorname{Im}(\theta) \geq 0, \theta \neq 2k\pi, \forall k \in \mathbb{Z}\}$ and we have

$$\limsup_{\operatorname{Im}(\lambda) > 0, \lambda \rightarrow 0} \|R_{\psi_1, \psi_2}(\lambda)\| < \infty.$$

Assuming conditions (H1) and (H2), we obtained the following estimates (see [9], [10]).

Theorem 2. Assume $n \geq 3$ and let $a(t, x)$ be a T -periodic metric satisfying (1.2) for which the conditions (H1) and (H2) are fulfilled. Let $2 \leq p, q < +\infty$ be such that

$$p > 2, \quad \frac{1}{p} = \frac{n(q-2)}{2q} - 1, \quad \frac{1}{p} \leq \frac{(n-1)(q-2)}{4q}. \quad (2.3)$$

Then for the solution $u(t)$ of (1.3) with $s = 0$ we have for all $t > 0$ the estimate

$$\|u(t)\|_{L^p(\mathbb{R}_t^+, L^q(R_x^n))} + \|u(t)\|_{\dot{H}^1(\mathbb{R}_x^n)} + \|\partial_t(u)(t)\|_{L^2(\mathbb{R}_x^n)} \leq C(p, q, \rho, T)(\|g_1\|_{\dot{H}^1(\mathbb{R}^n)} + \|g_2\|_{L^2(\mathbb{R}^n)}). \quad (2.4)$$

The crucial point in the proof of the global estimates (2.4) is the L^2 integrability with respect to t of the local energy (see [9], [10]). For this purpose we need to show that for cut-off functions $\psi_1, \psi_2 \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\psi_i = 1$ for $|x| \leq \rho + 1 + 3T, i = 1, 2$, for $t \geq s$ we have

$$\|\psi_1 \mathcal{U}(t, s) \psi_2\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \leq C_{\psi_1, \psi_2} d(t - s) \quad (2.5)$$

with $d(t) \in L^1(\mathbb{R}^+)$. To obtain (2.5), we use the assumption (H2). For $n \geq 3$, odd, we have an exponential decay of energy and $d(t) = e^{-\delta t}$, $\delta > 0$. For $n \geq 4$, even, we have another decay. In particular, the estimate

$$\|\psi_1 \mathcal{U}(NT, 0) \psi_2\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \leq \frac{C_{\psi_1, \psi_2}}{(N+1) \ln^2(N+e)}, \quad \forall N \in \mathbb{N}, \quad (2.6)$$

implies (2.5). On the other hand, if (2.6) holds, the assumption (H2) for n even is fulfilled. Indeed, for large $A \gg 1$ and $\text{Im}(\theta) \geq AT$ we have

$$R_{\psi_1, \psi_2}(\theta) = -e^{i\theta} \sum_{N=0}^{\infty} \psi_1 \mathcal{U}(NT, 0) \psi_2 e^{iN\theta}$$

and applying (2.6), we conclude that $R_{\psi_1, \psi_2}(\theta)$ admits a holomorphic extension from $\{\theta \in \mathbb{C} : \text{Im}(\theta) \geq A > 0\}$ to $\{\theta \in \mathbb{C} : \text{Im}(\theta) > 0\}$. Moreover, $R_{\psi_1, \psi_2}(\theta)$ is bounded for $\theta \in \mathbb{R}$. We refer to [10] for examples of metrics $a(t, x)$ such that (2.6) is fulfilled. We like to mention that in the study of the time-periodic perturbations of the Schrödinger operators (see [3]) the resolvent of the monodromy operator $(\mathcal{U}(T) - z)^{-1}$ plays a central role. Moreover, the absence of eigenvalues $z \in \mathbb{C}, |z| = 1$ of $\mathcal{U}(T)$, and the behavior of the resolvent for z near 1, are closely related to the decay of local energy as $t \rightarrow \infty$. So our results may be considered as a natural extension of those for Schrödinger operator. On the other hand, for the wave equation we may have poles $\theta \in \mathbb{C}, \text{Im}\theta > 0$ of the $R_{\psi_1, \psi_2}(\theta)$, while for the Schrödinger operator with time-periodic potentials such a phenomenon is excluded.

3. LOCAL TIME EXISTENCE

In this section we assume $n \geq 3$ and let $a(t, x)$ be a C^∞ function on \mathbb{R}^{n+1} satisfying the conditions (1.2). Motivated by the work of T. Tao and M. Keel in [12], we will apply Theorem 1 to find $k > 1$ for which the problem (1.1) is locally well-posed. For this purpose we need to find $k > 1$ so that there exist $2 \leq p, q < +\infty$ satisfying (2.1) with $\gamma = 1$ for which we have

$$k = \frac{q}{2}, \quad \frac{k}{p} < 1. \quad (3.1)$$

Then it is easy to see that $k > 1$ satisfies (3.1) with p, q satisfying (2.1), if the following conditions are fulfilled:

$$\begin{aligned} i) \quad & n = 3, \quad 3 < k < 5, \\ ii) \quad & n = 4, \quad 2 < k < 3, \\ iii) \quad & n = 5, \quad \frac{5}{3} < k < \frac{7}{3}, \\ iv) \quad & n \geq 6, \quad \frac{n}{n-2} < k \leq \frac{n}{n-3}. \end{aligned} \tag{3.2}$$

Now we recall a version of the Christ-Kiselev lemma.

Lemma 1. *Let X and Y be Banach spaces, and for all $s, t \in \mathbb{R}^+$ let $K(s, t)$ be an operator from X to Y . Suppose that*

$$\left\| \int_0^{t_0} K(s, t) h(s) ds \right\|_{L^l([t_0, +\infty[, Y)} \leq A \|h\|_{L^r(\mathbb{R}^+, X)},$$

for some $A > 0$, $1 \leq r < l \leq +\infty$, all $t_0 \in \mathbb{R}^+$ and $h \in L^r(\mathbb{R}^+, X)$. Then we have

$$\left\| \int_0^t K(s, t) h(s) ds \right\|_{L^l(\mathbb{R}^+, Y)} \leq AC_{r,l} \|h\|_{L^r(\mathbb{R}^+, X)},$$

where $C_{r,l} > 0$ depends only on r, l .

We refer to [7] for the proof of Lemma 1 (see also the original paper [1]). Notice that in [7] the above result is formulated with \mathbb{R} instead of \mathbb{R}^+ and $s, t, t_0 \in \mathbb{R}$, but, as it was mentioned in [7], the same proof works for intervals and in particular for \mathbb{R}^+ . We need the following

Lemma 2. *Let $a(t, x)$ satisfy the conditions (1.2). Let $T_1 \leq \delta$, and $2 \leq p, q < +\infty$ satisfy the conditions (2.1). Then for all $h \in L^1([0, T_1], L^2(\mathbb{R}^n))$ we have*

$$\left\| \int_0^t V(t, s) h(s) ds \right\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} \leq C \|h\|_{L^1([0, T_1], L^2(\mathbb{R}^n))} \tag{3.3}$$

with $C > 0$ independent of T_1 .

Proof. Let $t_0 \in [0, T_1]$. We have

$$\left\| \int_0^{t_0} V(t, s) h(s) ds \right\|_{L^p([t_0, T_1], L^q(\mathbb{R}^n))} \leq \int_0^{t_0} \|V(t, s) h(s)\|_{L^p([t_0, T_1], L^q(\mathbb{R}^n))} ds.$$

From the definition of $V(t, s)$ we know that

$$\begin{aligned} \|V(t, s) h(s)\|_{L^p([t_0, T_1], L^q(\mathbb{R}^n))} &= \|(\mathcal{U}(t, s)(0, h(s)))_1\|_{L^p([t_0, T_1], L^q(\mathbb{R}^n))} \\ &\leq \|(\mathcal{U}(t, 0)(\mathcal{U}(0, s)(0, h(s))))_1\|_{L^p([0, T_1], L^q(\mathbb{R}^n))}. \end{aligned}$$

Then, the estimate (2.2) implies that for all $s \in [0, t_0]$ we obtain

$$\begin{aligned} \|(\mathcal{U}(t, 0)(\mathcal{U}(0, s)(0, h(s))))_1\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} &\leq C_\delta \|\mathcal{U}(0, s)(0, h(s))\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} \\ &\leq C'_\delta \|h(s)\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where $C'_\delta = C_\delta \sup_{s \in [0, T_1]} \|\mathcal{U}(0, s)\|$ is independent of t_0 . It follows

$$\begin{aligned} \left\| \int_0^{t_0} V(t, s) h(s) ds \right\|_{L^p([t_0, T_1], L^q(\mathbb{R}^n))} &\leq C'_\delta \int_0^{t_0} \|h(s)\|_{L^2(\mathbb{R}^n)} ds \\ &\leq C'_\delta \|h\|_{L^1([0, T_1], L^2(\mathbb{R}^n))}. \end{aligned}$$

Consider $K(s, t) = \mathbb{1}_{[0, T_1]}(t) \mathbb{1}_{[0, T_1]}(s) V(t, s)$, $X = L^2(\mathbb{R}^n)$ and $Y = L^q(\mathbb{R}^n)$. Since $p > 1$, the Christ-Kiselev lemma yields

$$\left\| \int_0^t V(t, s) h(s) ds \right\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} \leq C(\delta, p) \|h\|_{L^1([0, T_1], L^2(\mathbb{R}^n))}.$$

□

Applying (3.3), we will show that problem (1.1) is locally well-posed for k and n satisfying the conditions (3.2).

Theorem 3. *Assume that $a(t, x)$ is a C^∞ function on \mathbb{R}^{n+1} satisfying conditions (1.2) and let k and n satisfy (3.2). Then there exists $T_1 > 0$ such that problem (1.1) admits a weak solution u on $[0, T_1]$. Moreover, u is the unique weak solution of (1.1) on $[0, T_1]$ satisfying the following properties:*

$$\begin{aligned} (i) \quad & u \in \mathcal{C}([0, T_1], \dot{H}^1(\mathbb{R}^n)), \quad (ii) \quad u_t \in \mathcal{C}([0, T_1], L^2(\mathbb{R}^n)), \\ (iii) \quad & u \in L^p([0, T_1], L^{2k}(\mathbb{R}^n)) \quad \text{with} \quad \frac{1}{p} = \frac{n(k-1)}{k} - 1. \end{aligned}$$

Proof. Let k and n satisfy (3.2). We have seen that we can find $2 \leq p, q < +\infty$ satisfying conditions (2.1) so that $\frac{k}{p} < 1$ and $\frac{k}{q} = \frac{1}{2}$. Consider the norm $\|\cdot\|_{Y_{T_1}}$ defined by

$$\|u\|_{Y_{T_1}} = \|u\|_{\mathcal{C}([0, T_1], \dot{H}^1)} + \|u\|_{L^p([0, T_1], L^q(\mathbb{R}^n))}$$

and

$$Y_{T_1} = \mathcal{C}([0, T_1], \dot{H}^1) \bigcap L^p([0, T_1], L^q(\mathbb{R}^n))$$

with T_1 to be determined. Notice that $(Y_{T_1}, \|\cdot\|_{Y_{T_1}})$ is a Banach space. Assume $f \in \dot{\mathcal{H}}_1(\mathbb{R}^n)$, $M > 0$ and let $B_M = \{u \in Y_{T_1} : \|u\|_{Y_{T_1}} \leq M\}$, with M to be determined. The problem of finding a weak solution u of (1.1) is equivalent to find a fixed point of the map

$$\mathcal{G}(u) = (\mathcal{U}(t, 0)g)_1 + \int_0^t V(t, s) f_k(u(s)) ds.$$

Let $u \in B_M$. We have

$$\begin{aligned} \left\| \int_0^t V(t, s) f_k(u(s)) ds \right\|_{\mathcal{C}([0, T_1], \dot{H}^1)} &\leq \sup_{t \in [0, T_1]} \int_0^t \mathbb{1}_{[0, t]}(s) \|V(t, s) f_k(u(s))\|_{\dot{H}^1} ds \\ &\leq \sup_{t \in [0, T_1]} \int_0^t \|V(t, s) f_k(u(s))\|_{\dot{H}^1} ds. \end{aligned}$$

The estimates (2.2) imply that for $T_1 \leq \delta$ there exists $C > 0$ independent of T_1 such that

$$\left\| \int_0^t V(t, s) f_k(u(s)) ds \right\|_{\mathcal{C}([0, T_1], \dot{H}^1)} \leq C \int_0^{T_1} \|f_k(u(s))\|_{L^2(\mathbb{R}^n)} ds \leq C_1 \int_0^{T_1} \| |u|^k(s) \|_{L^2(\mathbb{R}^n)} ds. \quad (3.4)$$

On the other hand, Lemma 2 yields

$$\left\| \int_0^t V(t, s) f_k(u(s)) ds \right\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} \leq C_2 \int_0^{T_1} \| |u|^k(s) \|_{L^2(\mathbb{R}^n)} ds. \quad (3.5)$$

We deduce from (3.4) and (3.5) that

$$\left\| \int_0^t V(t, s) f_k(u(s)) ds \right\|_{Y_{T_1}} \leq C_3 \int_0^{T_1} \| |u|^k(s) \|_{L^2(\mathbb{R}^n)} ds = C_3 \int_0^{T_1} \|u(s)\|_{L^q}^k ds. \quad (3.6)$$

Since $\frac{k}{p} < 1$, an application of the Hölder inequality yields

$$\left\| \int_0^t V(t, s) f_k(u(s)) ds \right\|_{Y_{T_1}} \leq C_3 \|u\|_{L^p([0, T_1], L^q(\mathbb{R}^n))}^k (T_1)^{1-\frac{k}{p}} \leq C_3 M^k (T_1)^{1-\frac{k}{p}}.$$

Let M be such that $\frac{M}{2} \geq 2C(\|g_1\|_{\dot{H}^1} + \|g_2\|_{L^2})$ and let T_1 be small enough such that

$$C_3 M^k (T_1)^{1-\frac{k}{p}} \leq \frac{M}{2}.$$

Then $\|\mathcal{G}(u)\|_{Y_{T_1}} \leq M$ and $\mathcal{G}(u) \in Y_{T_1}$. We have $\mathcal{G}(B_M) \subset B_M$ and B_M is a closed set of the Banach space $(Y_{T_1}, \|\cdot\|_{Y_{T_1}})$. Now we will show that we can choose T_1 small enough so that \mathcal{G} becomes a contraction. Let $u, v \in B_M$. We know that

$$\mathcal{G}(u) - \mathcal{G}(v) = \int_0^t V(t, s) (f_k(u(s)) - f_k(v(s))) ds.$$

In the same way as in the proof of inequality (3.4), Theorem 1 and Lemma 2 imply that

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{Y_{T_1}} \leq C_4 \int_0^{T_1} \|f_k(u(s)) - f_k(v(s))\|_{L^2} ds.$$

On the other hand, f_k satisfies

$$|f_k(u) - f_k(v)| \leq C_5 |u - v| (|u| + |v|)^{k-1}.$$

Consequently,

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{Y_{T_1}} \leq C_6 \int_0^{T_1} \|u(s) - v(s)\| (|u(s)| + |v(s)|)^{k-1} \|1\|_{L^2} ds.$$

Since $\frac{k-1}{q} + \frac{1}{q} = \frac{k}{q} = \frac{1}{2}$, by the generalized Hölder's inequality, we have

$$\begin{aligned} \| |u(s) - v(s)| (|u(s)| + |v(s)|)^{k-1} \|_{L^2} &\leq \|u - v\|_{L^q} \| (|u(s)| + |v(s)|)^{k-1} \|_{L^{\frac{q}{k-1}}} \\ &\leq \|u - v\|_{L^q} (\|u\|_{L^q} + \|v\|_{L^q})^{k-1}. \end{aligned}$$

This leads to

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{Y_{T_1}} \leq C_7 \int_0^{T_1} \|u(s) - v(s)\|_{L^q} (\|u(s)\|_{L^q} + \|v(s)\|_{L^q})^{k-1} ds.$$

Applying Hölder's inequality ones more, we find

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{Y_{T_1}} \leq C_7 (T_1)^{1-\frac{k}{p}} 2^{k-1} M^{k-1} \|u - v\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} \leq C_7 (T_1)^{1-\frac{k}{p}} (2M)^{k-1} \|u - v\|_{Y_{T_1}}.$$

Thus, if we choose T_1 so that

$$C_7 (2M)^{k-1} (T_1)^{1-\frac{k}{p}} < 1,$$

\mathcal{G} will be a contraction from B_M to B_M . Consequently, there exists a unique $u \in Y_{T_1}$ such that

$$\|u\|_{Y_{T_1}} \leq M \quad \text{and} \quad \mathcal{G}(u) = u.$$

□

Remark 2. In contrast to the case $a = 1$ (see [12], [13] and [15]) for our argument we must use homogeneous Strichartz estimates. This restriction leads to a solution in the energy space $\dot{\mathcal{H}}_1(\mathbb{R}^n)$. Moreover, we have more restrictions on the values of $k > 1$.

Since we use estimates (2.2) to prove Theorem 3, the length T_1 of the interval $[0, T_1]$ on which the existence result holds, is majored by the length δ of the interval on which estimates (2.2) are established. To improve this existence result, in the same way as in [12], we will apply global estimates in the next section.

4. LONG TIME EXISTENCE FOR SMALL INITIAL DATA

In this section we assume that $n \geq 3$, $a(t, x)$ is T -periodic with respect to t and (H1), (H2) are fulfilled. We will use the estimates (2.4) to find solutions of (1.1) defined in $[0, T_1]$, with T_1 only depending on k , n and g . For this purpose we must find $k > 1$ such that there exist $2 \leq p, q < +\infty$ satisfying (2.3) for which

$$k = \frac{q}{2}, \quad \frac{k}{p} < 1. \quad (4.1)$$

Then $k > 1$ satisfies (4.1) with p, q satisfying (2.3) if the following conditions are fulfilled

$$\begin{aligned} i) \quad & n = 3, \quad 3 < k < 5, \\ ii) \quad & n = 4, \quad 2 < k < 3, \\ iii) \quad & n = 5, \quad \frac{5}{3} < k < \frac{7}{3}, \\ iv) \quad & n \geq 6, \quad \frac{n}{n-2} < k < \frac{n}{n-3}. \end{aligned} \quad (4.2)$$

Lemma 3. *Assume that (H1) and (H2) are fulfilled, $a(t, x)$ is T -periodic with respect to t and $n \geq 3$. Let $t \geq s \geq 0$. Then*

$$\|\mathcal{U}(t, s)\|_{\mathcal{L}(\dot{\mathcal{H}}_1(\mathbb{R}^n))} \leq C_0$$

with $C_0 > 0$ independent of s and t .

Proof. Let $m \in \mathbb{N}$ be such that $0 \leq s - mT < T$. We have

$$\mathcal{U}(t, s) = \mathcal{U}(t - mT, s - mT) = \mathcal{U}(t - mT, 0)\mathcal{U}(0, s - mT).$$

Since $t - mT \geq s - mT \geq 0$, Theorem 2 implies

$$\|\mathcal{U}(t - mT, 0)\|_{\mathcal{L}(\dot{\mathcal{H}}_1(\mathbb{R}^n))} \leq C'$$

with $C' > 0$ independent of t . Also we have

$$\|\mathcal{U}(0, s - mT)\|_{\mathcal{L}(\dot{\mathcal{H}}_1(\mathbb{R}^n))} \leq \sup_{s' \in [0, T]} \|\mathcal{U}(0, s')\|_{\mathcal{L}(\dot{\mathcal{H}}_1(\mathbb{R}^n))} = C''.$$

It follows that

$$\|\mathcal{U}(t, s)\|_{\mathcal{L}(\dot{\mathcal{H}}_1(\mathbb{R}^n))} \leq C' C'' = C_0$$

and C_0 is independent of t and s . □

The estimates (2.4), the Christ-Kiselev lemma and Lemma 3 imply the following

Lemma 4. *Assume $n \geq 3$ and let $a(t, x)$ be T -periodic with respect to t such that (H1) and (H2) are fulfilled. Let $2 \leq p, q < +\infty$ satisfy condition (2.3) and let $T_1 > 0$. Then for all $h \in L^1([0, T_1], L^2(\mathbb{R}^n))$ we have*

$$\left\| \int_0^t V(t, s)h(s)ds \right\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} \leq C \|h\|_{L^1([0, T_1], L^2(\mathbb{R}^n))}$$

with $C > 0$ independent of g and T_1 .

Proof. Let $t_0 > 0$, $s \in [0, t_0]$ and $t > t_0$. Consider $mT \leq t_0 < (m+1)T$. We have

$$\begin{aligned} V(t, s)h(s) &= (\mathcal{U}(t - mT, s - mT)(0, h(s)))_1 \\ &= (\mathcal{U}(t - mT, 0)\mathcal{U}(0, s - mT)(0, h(s)))_1 \\ &= (\mathcal{U}(t - mT, 0)\mathcal{U}(mT, s)(0, h(s)))_1. \end{aligned}$$

Thus, the estimate (2.4) implies

$$\left\| \int_0^{t_0} V(t, s)h(s)ds \right\|_{L^p([t_0, +\infty[, L^q(\mathbb{R}^n))} \leq C \int_0^{t_0} \|\mathcal{U}(mT, s)(0, h(s))\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} ds.$$

Since $0 \leq s \leq mT$ for $s \in [0, mT]$, Lemma 3 yields

$$\int_0^{mT} \|\mathcal{U}(mT, s)(0, h(s))\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} ds \leq C_0 \int_0^{mT} \|h(s)\|_{L^2(\mathbb{R}^n)} ds$$

with $C > 0$ independent of t_0 . In the same way, since $mT \leq t_0 < (m+1)T$ we have

$$\int_{mT}^{t_0} \|\mathcal{U}(0, s - mT)(0, h(s))\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} ds \leq \sup_{s \in [0, T]} \|\mathcal{U}(0, s)\| \int_{mT}^{t_0} \|h(s)\|_{L^2(\mathbb{R}^n)} ds.$$

It follows that

$$\left\| \int_0^{t_0} V(t, s)h(s)ds \right\|_{L^p([t_0, +\infty[, L^q(\mathbb{R}^n))} \leq C \int_0^{+\infty} \|h(s)\|_{L^2(\mathbb{R}^n)} ds.$$

Since $p > 1$, the Christ- Kiselev lemma implies

$$\left\| \int_0^t V(t, s)h(s)ds \right\|_{L^p(\mathbb{R}^+, L^q(\mathbb{R}^n))} \leq C_p \|h\|_{L^1(\mathbb{R}^+, L^2(\mathbb{R}^n))}.$$

We deduce that

$$\begin{aligned} \left\| \int_0^t V(t, s)h(s)ds \right\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} &\leq \left\| \int_0^t V(t, s)\mathbf{1}_{[0, T_1]}(s)h(s)ds \right\|_{L^p(\mathbb{R}^+, L^q(\mathbb{R}^n))} \\ &\leq C \|h\|_{L^1([0, T_1], L^2(\mathbb{R}^n))} \end{aligned}$$

with $C > 0$ independent of T_1 . □

Theorem 4. Assume that k and n satisfy the conditions (4.2). Let $a(t, x)$ be T -periodic with respect to t and let (H1), (H2) be fulfilled. Then there exists $C(k, f_k, T, \rho, n)$ such that for all $g \in \dot{\mathcal{H}}_1(\mathbb{R}^n)$ we can find a weak solution u of (1.1) on $[0, T_1]$ with

$$T_1 = C(k, f_k) \left(\|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} \right)^{-d}, \quad (4.3)$$

where $d = \frac{2(k-1)}{(n+2) - (n-2)k}$. Moreover, u is the unique weak solution of (1.1) on $[0, T_1]$ satisfying the following properties:

$$\begin{aligned} (i) \quad &u \in \mathcal{C}([0, T_1], \dot{H}^1(\mathbb{R}^n)), \quad (ii) \quad u_t \in \mathcal{C}([0, T_1], L^2(\mathbb{R}^n)), \\ (iii) \quad &u \in L^p([0, T_1], L^{2k}(\mathbb{R}^n)) \quad \text{with} \quad \frac{1}{p} = \frac{n(k-1)}{k} - 1. \end{aligned} \quad (4.4)$$

Proof. Let $C_f > 0$ be such that $|f_k(u)| \leq C_f |u|^k$ and $|f_k(u) - f_k(v)| \leq C_f |u - v| (|u| + |v|)^{k-1}$. Then, Theorem 2 and Lemma 4 imply that there exists A_k such that for all $T_1 > 0$

$$\left\| \int_0^t V(t, s)h(s)ds \right\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} \leq A_k \|h\|_{L^1([0, T_1], L^2(\mathbb{R}^n))}$$

and

$$\|(\mathcal{U}(t, 0)g)_1\|_{L^p([0, T_1], L^q(\mathbb{R}^n))} \leq A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}.$$

According to the proof of Theorem 3, $\mathcal{G}(u) = (\mathcal{U}(t, 0)g)_1 + \int_0^t V(t, s)f_k(u(s))ds$ admits a fixed point in the set

$$\{u \in \mathcal{C}([0, T_1], \dot{H}^1) \cap L^p([0, T_1], L^q) : \|u\|_{\mathcal{C}([0, T_1], \dot{H}^1)} + \|u\|_{L^p([0, T_1], L^q)} \leq M\}$$

if we choose $M, T_1 > 0$ so that

$$\begin{cases} A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} + C_3 M^k (T_1)^{1-\frac{k}{p}} \leq M, \\ C_7 (2M)^{k-1} (T_1)^{1-\frac{k}{p}} < 1. \end{cases} \quad (4.5)$$

In particular (4.5) will be fulfilled if

$$\begin{cases} A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} + C_3 M^k (T_1)^{1-\frac{k}{p}} = M, \\ C_7 (2M)^{k-1} (T_1)^{1-\frac{k}{p}} < 1. \end{cases} \quad (4.6)$$

We will choose M, T_1 so that (4.6) holds. Let $t_1 = (T_1)^{1-\frac{k}{p}}$. We find that the system (4.6) is equivalent to the following

$$\begin{cases} t_1 = \frac{M - A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}}{C_3 M^k}, \\ 0 < \frac{M - A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}}{M} < \frac{1}{C_7 2^{k-1}}. \end{cases} \quad (4.7)$$

Since $M \mapsto \frac{M - A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}}{M}$ is strictly increasing, we obtain that (t_1, M) is a solution of (4.7) if

$$M < \frac{2^{k-1} A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}}{2^{k-1} - 1}.$$

Take

$$M_0 = \frac{\alpha 2^{k-1} A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}}{2^{k-1} - 1} \quad \text{and} \quad t_1 = \frac{M_0 - A_k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}}{C_3 (M_0)^k}$$

with $1 - \frac{1}{2^{k-1}} < \alpha < 1$. Then (M_0, t_1) is a solution of (4.7) and we have

$$t_1 = \frac{\frac{\alpha 2^{k-1}}{2^{k-1}-1} - 1}{C_8 \left(\frac{2^{k-2} A_k}{2^{k-1}-1} \right)^k \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}^{k-1}} = C'(k, f_k) \|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}^{-(k-1)}.$$

Thus for $M = M_0$ and

$$T_1 = (t_1)^{\frac{1}{1-\frac{k}{p}}} = C(k, f_k) \left(\|g\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} \right)^{-\frac{k-1}{1-\frac{k}{p}}},$$

M and T_1 satisfy conditions (4.6). Moreover, we know that

$$\frac{k}{p} = \frac{(n-2)k - n}{2}.$$

Thus we have

$$\frac{k-1}{1-\frac{k}{p}} = \frac{2(k-1)}{(n+2)-(n-2)k}$$

and M, T_1 satisfy conditions (4.5) if $M = M_0$ and

$$T_1 = C(k, f_k) \left(\|g\|_{\dot{H}^1(\mathbb{R}^n)} \right)^{-\frac{2(k-1)}{(n+2)-(n-2)k}}.$$

Note that for $n \geq 6$ we have $\frac{n}{n-3} \leq \frac{n+2}{n-2}$ and $k < \frac{n}{n-3}$ leads to $k < \frac{n+2}{n-2}$. □

Remark 3. Let $\|g\|_{\dot{H}^1(\mathbb{R}^n)} = \epsilon$ and $T_1 = C\epsilon^{-d}$, $C, d > 0$ being the constants defined by (4.3). Then Theorem 4 implies that there exists a unique solution of (1.1) satisfying (4.4).

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